Copolymer networks and stars: Scaling exponents

C. von Ferber[1,2] and Yu. Holovatch[3]
[1]*School of Physics and Astronomy, Tel Aviv University, IL-69978 Tel Aviv, Israel*
[2]*Fachbereich Physik, Universität-Gesamthochschule-Essen, D-45117 Essen, Germany*
[3]*Institute for Condensed Matter Physics, Ukrainian Academy of Sciences, UA-290011 Lviv, Ukraine*
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We explore and calculate the rich scaling behavior of copolymer networks in solution by renormalization-group methods. We establish a field-theoretic description in terms of composite operators. Our third-order resummation of the spectrum of scaling dimensions brings about remarkable features: The special convexity properties of the spectra allow for a multifractal interpretation while preserving stability of the theory. This behavior could not be found for power-of-field operators of usual ϕ^4 field theory. The two-dimensional (2D) limit of the mutually avoiding walk star apparently corresponds to results of a conformal Kac series. Such a classification seems not possible for the 2D limit of other copolymer stars. We furthermore provide a consistency check of two complementary renormalization schemes: ε expansion and renormalization at fixed dimension, calculating a large collection of independent exponents in both approaches. [S1063-651X(97)08911-3]

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I. INTRODUCTION

Recently, there has been considerable interest in the relation of field theory and multifractals [1–4] and the associated multifractal dimension spectra [5–10] as well as nonintersecting random walks and their two-dimensional (2D) conformal theory [11–15]. We present a model of multicomponent polymer networks that shows a common core of these topics and allows for a detailed study of the interrelations. While a multifractal spectrum may be derived from the scaling exponents of two mutually avoiding stars of random walks [1,5], a description in terms of power-of-field operators seems to be ruled out by stability considerations [1]. Here we show that already a simple product of two power-of-field operators complies with both somewhat contrary requirements. We start from the theory of polymer stars, starlike arrangements of polymer chains with self- and mutual excluded interactions [16,17]. We generalize this concept to stars of chains of different polymer species that may differ in their self- and mutual interactions. Our formalism describes homogeneous polymer stars, stars of mutually avoiding walks, and the situation of two mutually interacting stars.

Polymers and polymer solutions are among the most intensively studied objects in condensed matter physics [18]. The behavior of multicomponent solutions containing polymers of different species is especially rich. Systems of chemically linked polymer chains of different species such as block copolymers are also of considerable experimental and technical interest. Linking polymer chains of different or even contrary properties, such as hydrophile and hydrophobe chains, one obtains systems with qualitatively different behavior. Here we concentrate on systems of chains in a solvent that, in a given temperature range, differ in their respective steric interaction properties. We generalize the theory of multicomponent polymer solutions [19–23] to solutions of copolymer networks, i.e., chains of different species linked at their end points in the form of stars or networks of any topology (see Figs.1–3).

For solutions of polymer networks of a single species the scaling properties have been extensively studied by renormalization-group methods (for a review see [24]). Star polymers as the most simple polymer networks may be produced by linking together the end points of polymer chains at some core molecule (Fig. 1).

Networks of any given topology may be generated in the same way (Fig. 2). Randomly linked polymer networks are also obtained as a result of a vulcanization process by randomly linking nearby monomers of different chains to each other.

The asymptotic properties of homogeneous systems of linear chain molecules in solution are universal in the limit of long chains. Let us give a short account of the standard textbook results [18,25–28]. For each system one finds a so-called Θ temperature at which the two point attractive and repulsive interactions between the different monomers compensate for each other. As a result, the polymer chains may be described by random walks (up to higher-order correc-

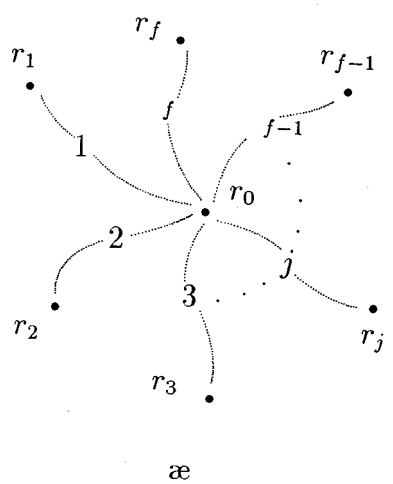

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FIG. 1. Star polymer of f arms linked together at point r_0 with extremities placed at points $r_1, \ldots, r_f$.

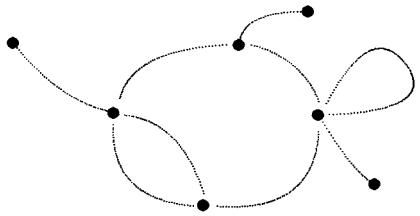

FIG. 2. Polymer network $\mathcal{G}$. It is characterized by the numbers n_f of f-leg vertices. Here $n_1=3$, $n_3=2$, $n_4=1$, and $n_5=1$.

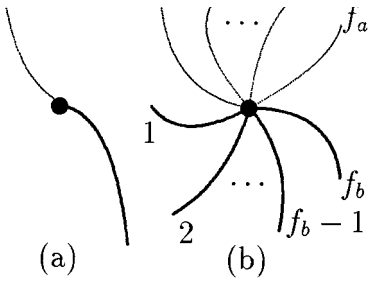

FIG. 3. (a) Block copolymer consisting of two polymer chains of different species (shown by solid and thin lines) linked at their end points. (b) Copolymer star consisting of f_a arms of species a and f_b arms of species b tied together at their end points.

tions): The mean-square distance between the chains' end points $\langle R^2\rangle$ scales with the number of monomers N like $\langle R^2\rangle\sim N$. Above the Θ temperature the effective interaction between the monomers is repulsive resulting in a swelling of the polymer coil that is universal for $N\to\infty$:

$$\langle R^2\rangle\sim N^{2\nu}, \tag{1}$$

where the correlation length exponent $\nu=0.588$, 3/4 in space dimensions $d=3,2$. The number of configurations $\mathcal{Z}$ of a polymer chain of N monomers grows for $N\to\infty$ like

$$\mathcal{Z}\sim e^{\mu N}N^{\gamma-1}, \tag{2}$$

with a nonuniversal fugacity e^{μ} and a universal exponent $\gamma=1.160$, 43/32 for $d=3,2$. In the early 1970s, following the work of de Gennes [29], the scaling theory of polymers was elaborated in detail using the analogy between the asymptotic properties of long polymer chains and the long-distance correlations of a magnetic system in the vicinity of the second-order phase transition (see [18,26]). This mapping allows one to receive the above-defined exponents ν and γ as limits of the correlation length exponent ν and the magnetic susceptibility critical exponent γ of the $O(m)$ symmetric m-vector model in the formal limit $m\to 0$ [29].

On the other hand, if polymers of different species are present in the same solution, the scaling behavior of the observables may be much more rich. Let us consider a solution of two different species of polymers in some solvent, a so-called ternary solution. Depending on the temperature, the system may then behave as if one or more of the inter- and intrachain interactions vanish in the sense described above [19–23]. This will lead to asymptotic scaling laws that may differ from those observed for each species alone [30].

Interesting systems are obtained when linking together polymers of different species. The most simple system of this kind is a so-called block copolymer consisting of two parts of different species. They are of some technical importance, e.g., serving as surfactants [31]. For our study they give the most simple example of a polymer star consisting of chains of two different species [Fig. 3(a)], which we will call here a copolymer star. For the homogeneous polymer star the asymptotic properties are uniquely determined by the number of its constituting chains and the dimension of space. For the number of configurations (partition function) $\mathcal{Z}_f$ of a polymer star of f chains each consisting of N monomers one finds [17,24]

$$\mathcal{Z}_f\sim e^{\mu N f}N^{\gamma_f-1}\sim(R/\ell)^{\eta_f-f\eta_2},\quad N\to\infty. \tag{3}$$

The second part shows scaling with the size $R\sim N^{\nu}$ of the isolated coil of N monomers on some scale ℓ, omitting the fugacity factor. The exponents γ_f,η_f, $f=1,2,3,\ldots$, constitute families of ''star exponents,'' which depend on the number of arms f in a nontrivial way. The case of linear polymer chains is included in this family with the exponent $\gamma=\gamma_1=\gamma_2$ defined in Eq. (2). For general numbers of arms f the star exponents γ_f,η_f have no physical counterparts in the set of exponents describing magnetic phase transitions. Nevertheless, they can be related to the scaling dimensions of composite operators of traceless symmetry in the polymer limit $m\to 0$ of the $O(m)$ symmetric m-vector model [32,33]. The exponents γ_f have been calculated analytically in perturbation theory [16,17,32–34], by exact methods in two dimensions [17,35], and by Monte Carlo simulations [36,37].

It has been shown that the scaling properties of polymer networks of arbitrary but fixed topology are uniquely defined by its constituting stars [24], as long as the statistical ensemble respects some conditions on the chain length distribution [32]. Thus the knowledge of the set of star exponents γ_f or η_f allows one to obtain the power laws corresponding to Eq. (3) also for any polymer network of arbitrary topology. The partition function $\mathcal{Z}_{\mathcal{G}}$ of a polymer network $\mathcal{G}$ (see Fig. 2) of F chains each of N monomers scales with $N\to\infty$ according to [17]

$$\mathcal{Z}_{\mathcal{G}}\sim e^{\mu FN}N^{\nu(\eta_{\mathcal{G}}-F\eta_2)}\quad\text{with }\eta_{\mathcal{G}}=-dL+\sum_{f\geq 1}n_f\eta_f, \tag{4}$$

where n_f is the number of vertices with f legs and $L=1+\Sigma(f/2-1)n_f$ is the number of loops in the network $\mathcal{G}$.

In this article we address a somewhat more complex problem: What happens to the scaling laws if we build a polymer star or general network of chains of different species? In view of the above-introduced ternary solutions, we thus study systems of polymer networks in which some of the intra- and interchain interactions may vanish. For instance, there may be *only mutual* excluded-volume interactions between chains of two different species in the copolymer star shown in Fig. 3, while chains of the same species may freely intersect. Each subset of chains of one species thus constitutes a star of random walks avoiding the second star of random walks. Cates and Witten [5] have shown that this problem of a star of random walks avoiding a given fractal

structure is the key to calculating the multifractal spectrum of absorption of diffusing particles on this fractal. This calculation can be performed explicitly for the diffusion near absorbing polymers [5,38].

The setup of our article is as follows. In Sec. II we introduce notation and relate the polymer model to a Lagrangian field theory. This field-theoretical formalism will be used throughout the paper. In Sec. III we define the renormalization-group procedures. We present two alternative approaches: zero mass renormalization together with an ε expansion (see e.g., [39]) and massive renormalization at a fixed dimension [40]. Section IV is devoted to the study of the renormalization-group flow of the ternary model and its fixed points. Series for critical exponents governing the scaling behavior of copolymer stars and stars of mutually avoiding walks are obtained in Sec. V. In Sec. VI we discuss the problem of resummation of the asymptotic series arising in this context. Numerical results are presented in Sec. VII. We close with concluding remarks and an outlook on possible applications of the theory in Sec. VIII and give some calculational details in Appendixes. We have announced some of our main results in a Letter [41].

II. MODEL AND NOTATIONS

Let us first take a look at the model we use to describe polymers. In a first discrete version we describe a configuration $\mathcal{C}$ of the polymer by a set of positions of segment end points:

$$\mathcal{C}\{r_1,\ldots,r_N\}\in\mathbb{R}^{d\times N}.$$

Its statistical weight (Boltzmann factor) with the Hamiltonian $\mathcal{H}$ divided by the product of the Boltzmann constant k_B and temperature T will be given by

$$\exp\left[-\frac{1}{k_BT}\mathcal{H}\right]=\exp\left\{-\frac{1}{4\ell_0^2}\sum_{j=1}^{N}(r_j-r_{j-1})^2-\beta\ell_0^d\sum_{i\neq j=1}^{N}\delta^d(r_i-r_j)\right\}. \tag{5}$$

The first term describes the chain connectivity and the parameter ℓ_0 governs the mean-sqare segment length. The second term describes the excluded-volume interaction forbidding two segment end points to take the same position in space. The parameter β gives the strength of this interaction. The third parameter in our model is the chain length or number of segments N. The partition function $\mathcal{Z}$ is calculated as an integral over all configurations of the polymer divided by the system volume Ω, thus dividing out identical configurations just translated in space:

$$\mathcal{Z}(N)=\frac{1}{\Omega}\int\prod_{i=1}^{N}dr_i\exp\left[-\frac{1}{k_BT}\mathcal{H}\{r_i\}\right]. \tag{6}$$

This will give us the ''number of configurations'' of the polymer (2). We will do our investigations by mapping the polymer model to a renormalizable field theory making use of well-developed formalisms (see [18,26] for example). To this end we introduce a continuous version of our model as proposed by Edwards [42,43], generalizing it to describe a set of f polymer chains of varying composition possibly tied together at their end points. The configuration of one polymer is now given by a path $r^a(s)$ in d-dimensional space $\mathbb{R}^d$ parametrized by a surface variable $0\leqslant s\leqslant S_a$. We now allow for a symmetric matrix of excluded-volume interactions u_{ab} between chains $a,b=1,\ldots,f$. The Hamiltonian $\mathcal{H}$ is then given by

$$\frac{1}{k_BT}\mathcal{H}(r^a)=\sum_{a=1}^{f}\int_0^{S_a}ds\left(\frac{dr^a(s)}{2ds}\right)^2+\frac{1}{6}\sum_{a,b=1}^{f}u_{ab}\int d^dr\;\rho_a(r)\rho_b(r), \tag{7}$$

with densities $\rho_a(r)=\int_0^{S_a}ds\;\delta^d(r-r^a(s))$. In this formalism the partition function is calculated as a functional integral

$$\mathcal{Z}_f\{S_a\}=\int\mathcal{D}[r^a(s)]\exp\left\{-\frac{1}{k_BT}\mathcal{H}(r^a)\right\}. \tag{8}$$

Here the symbol $\mathcal{D}[r_a(s)]$ includes normalization such that $Z\{S_a\}=1$ for all $u_{ab}=0$. To make the exponential of δ functions in Eq. (8) and the functional integral well defined in the bare theory a cutoff s_0 is introduced such that all simultaneous integrals of any variables s and s' on the same chain are cut off by $|s-s'|>s_0$. Let us note here that the continuous chain model (7) may be understood as a limit of discrete self-avoiding walks, when the length of each step is decreasing $\ell_0\to 0$ while the number of steps N_a is increasing, keeping the ''Gaussian surface'' $S_a=N_a\ell_0^2$ fixed. The continuous chain model (8) can be mapped onto a corresponding field theory by a Laplace transform in the Gaussian surface variables S_a to conjugate chemical potentials (''mass variables'') μ_a [30]:

$$\tilde{\mathcal{Z}}_f\{\mu_a\}=\int_0^{\infty}\prod_b\;dS_b e^{-\mu_bS_b}\mathcal{Z}_f\{S_a\}. \tag{9}$$

The Laplace-transformed partition function $\tilde{\mathcal{Z}}_f\{\mu_a\}$ can be expressed as the $m=0$ limit of the functional integral over vector fields ϕ_a, $a=1,\ldots,f$, with m components ϕ_a^α, $\alpha=1,\ldots,m$:

$$\tilde{\mathcal{Z}}_f\{\mu_b\}=\int\mathcal{D}[\phi_a(r)]\exp[-\mathcal{L}\{\phi_b,\mu_b\}]\;|_{m=0}. \tag{10}$$

The Landau-Ginzburg-Wilson Lagrangian $\mathcal{L}$ of f interacting fields ϕ_b, each with m components, reads

$$\mathcal{L}\{\phi_b,\mu_b\}=\frac{1}{2}\sum_{a=1}^{f}\int d^dr\{\mu_a\phi_a^2+[\nabla\phi_a(r)]^2\}+\frac{1}{4!}\sum_{a,a'=1}^{f}u_{a,a'}\int d^dr\ \phi_a^2(r)\phi_{a'}^2(r), \tag{11}$$

where $\phi_a^2=\Sigma_{\alpha=1}^{m}(\phi_a^\alpha)^2$. The limit $m=0$ in Eq. (10) can be understood as a selection rule for the diagrams contributing to the perturbation theory expansions, which can be easily checked diagrammatically. A formal proof of Eq. (11) using the Stratonovich-Hubbard transformation to linearize terms in Eq. (7) is given for the multicomponent case in [30]. The one-particle irreducible vertex functions $\Gamma^{(L)}(q_i)$ of this theory are defined by

$$\delta\Big(\sum\ q_i\Big)\Gamma^{(L)}_{a_1,\ldots,a_L}(q_i)=\int e^{iq_ir_i}dr_1\cdots dr_L\langle\phi_{a_1}(r_1)\cdots\phi_{a_L}(r_L)\rangle^{\mathcal{L}}_{1\mathrm{PI},m=0}. \tag{12}$$

The average $\langle\ \rangle$ in Eq. (12) is understood with respect to the Lagrangian (11) keeping only those contributions that correspond to one-particle irreducible graphs and have nonvanishing tensor factors in the limit $m=0$. The partition function $Z_{*f}\{S_a\}$ of a polymer star consisting of f polymers of different species $1,\ldots,f$ constrained to have a common end point is obtained from Eq. (8) by introducing an appropriate product of δ functions ensuring the ''starlike'' structure. It reads

$$Z_{*f}\{S_a\}=\int\mathcal{D}[r_a]\exp\Big\{-\frac{1}{k_BT}\mathcal{H}(r_a)\Big\}\prod_{a=2}^{f}\delta^d(\vec{r}_a(0)-\vec{r}_1(0)). \tag{13}$$

The vertex part of its Laplace transformation may be defined by

$$\delta\Big(p+\sum\ q_i\Big)\Gamma^{(*f)}(p,q_1,\ldots,q_f)=\int e^{i(pr_0+q_ir_i)}d^dr_0d^dr_1\cdots d^dr_f\langle\phi_1(r_0)\cdots\phi_f(r_0)\phi_1(r_1)\cdots\phi_f(r_f)\rangle^{\mathcal{L}}_{1\mathrm{PI},m=0}, \tag{14}$$

where all $a_1,\ldots,a_f$ are distinct. The vertex function $\Gamma^{(*f)}$ is thus defined by insertion of the composite operator $\Pi_a\phi_a$. Its scaling properties are governed by the scaling dimension of this operator. When only one species is present one can also define Γ^{*f} by insertion of a composite operator of traceless symmetry [33]. In the following we will be mainly interested in the case of only two species of polymers, with interactions u_{11},u_{22} between the polymers of the same species and $u_{12}=u_{21}$ between the polymers of different species. In this case the composite operator in Eq. (14) reduces to the product of two power-of-field operators with appropriate symmetry $(\phi)^{f_1}(\phi')^{f_2}$, each corresponding to a product of fields of the same ''species.'' Nevertheless, our results are easily generalized to the case of any number of polymer species.

The starting point for our calculations are the three-loop expansions for the bare vertex functions of interest $(\partial/\partial k^2)\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(*f)}$. They involve the loop integrals D_2 and I_1-I_8. These are given in Appendix A together with their corresponding graphs. The expressions read

$$\frac{\partial}{\partial k^2}\Gamma^{(2)}_{(aa)}=1-\frac{1}{9}I_2u_{aa}^2+\frac{4}{27}I_8u_{aa}^3, \tag{15}$$

$$\Gamma^{(4)}_{(aaaa)}=u_{aa}-\frac{4}{3}D_2^{aa}u_{aa}^2+\left(\frac{5}{9}D_2^2+\frac{22}{9}I_1\right)u_{aa}^3-\left(\frac{2}{9}D_2^3+\frac{28}{27}I_1D_2+\frac{8}{27}I_3+\frac{40}{9}I_4+\frac{58}{27}I_5+\frac{14}{27}I_6+\frac{22}{27}I_7\right)u_{aa}^4, \tag{16}$$

$$\begin{aligned}\Gamma^{(*f)}=1&+D_2^{a_1a_2}\overline{u}_{a_1a_2}/2+D_2^2\overline{u}_{a_1a_2}\overline{u}_{a_3a_4}/8+\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}I_1+\overline{u}_{a_1a_1}\overline{u}_{a_1a_2}I_1+(I_1+D_2^2)\overline{u}^2_{a_1a_2}/2+D_2^3\overline{u}_{a_1a_2}\overline{u}_{a_3a_4}\overline{u}_{a_5a_6}/48\\&+D_2\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}\overline{u}_{a_4a_5}I_1/2+\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}\overline{u}_{a_3a_4}I_4+(I_5+I_6)\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}\overline{u}_{a_2a_4}/2+\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}\overline{u}_{a_1a_4}I_4+(3I_4+I_7)\\&\times\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}\overline{u}_{a_2a_3}/3+D_2(I_1+D_2^2)\overline{u}^2_{a_1a_2}\overline{u}_{a_3a_4}/4+(D_2I_1+2I_4+2I_5+I_7)\overline{u}^2_{a_1a_2}\overline{u}_{a_1a_3}+(I_3+3I_4+I_5)\\&\times\overline{u}^2_{a_1a_1}\overline{u}_{a_1a_2}+(D_2I_1+2I_4+I_5+I_6+D_2^3)\overline{u}^3_{a_1a_2}/2+D_2\overline{u}_{a_1a_1}\overline{u}_{a_1a_2}\overline{u}_{a_3a_4}I_1/2+(I_4+I_5)\overline{u}_{a_1a_1}\overline{u}_{a_1a_2}\overline{u}_{a_2a_3}\\&+(I_4+I_5+I_7)\overline{u}_{a_1a_1}\overline{u}_{a_1a_2}\overline{u}_{a_1a_3}+\overline{u}_{a_1a_1}\overline{u}_{a_1a_2}\overline{u}_{a_2a_2}I_5/2+(D_2I_1+4I_4+I_6)\overline{u}_{a_1a_1}\overline{u}^2_{a_1a_2}.\end{aligned} \tag{17}$$

In Eq. (17) summation over $a_i=1,\ldots,f$ is assumed. Equations (15)–(17) apply to any number of polymer species. For a star of f_1 chains of species 1 and f_2 chains of species 2 we restrict $a=1,2$ in Eqs. (15) and (16) and the matrix of interactions $\overline{u}_{ab}$ is given by

$$\overline{u}_{ab}^{f_1 f_2}=\begin{cases} u_{11}, & 1\leqslant a,b\leqslant f_1 \\ u_{22}, & f_1<a,b\leqslant f \\ u_{12}, & \text{otherwise.} \end{cases}$$

Let us define in this way

$$\Gamma^{(*f_1 f_2)}=\Gamma^{(*f)}\big|_{\overline{u}_{ab}=\overline{u}_{ab}^{f_1 f_2}}. \tag{18}$$

For general f_1,f_2 the corresponding combinatorics may also be directly calculated by summation over $a_i=1,2$ instead. Replacing $\overline{u}_{a_i a_j}=u_{a_i a_j}$, each term in the sum with indices $a_1,\ldots,a_k$ then acquires a factor

$$\binom{f_1}{\mathcal{N}_1(a_1,\ldots,a_k)}\binom{f_2}{\mathcal{N}_2(a_1,\ldots,a_k)}.$$

Here $\mathcal{N}_1(a_1,\ldots,a_k)$ is the number of $a_i=1$, whereas $\mathcal{N}_2(a_1,\ldots,a_k)$ is the number of $a_i=2$.

As a special case we may derive the vertex function $\Gamma^{(4)}_{1122}$ for the u_{12} interaction using the relation $\Gamma^{(*22)}=\partial/\partial u_{12}\Gamma^{(4)}_{1122}$, which is obvious from the perturbation theory (see [30] for instance):

$$\Gamma^{(4)}_{1122}=\int du_{12}\Gamma^{(*22)}.$$

Note that the vertex functions $\Gamma^{(*20)}$ and $\Gamma^{(*11)}$ define a vertex function with a ϕ^2 insertion which in standard literature is denoted by $\Gamma^{(2,1)}$ [39]. With the same formalism we can also describe a star of f mutually avoiding walks [11,12]. In this case all interactions on the same chain $\overline{u}_{aa}$ vanish and only those $\overline{u}_{ab}$ with $a\neq b$ remain:

$$\Gamma^{(*f)}_{\mathrm{MAW}}=\Gamma^{(*f)}\big|_{\overline{u}_{ab}=(1-\delta_{ab})u_{12}}. \tag{19}$$

In this case each term with indices $a_1,\ldots,a_k$ acquires a factor $(f/k)k!$.

As is well known, ultraviolet divergences occur when the vertex functions (15)–(17) are evaluated naively [44]. In the next section we apply the field-theoretic renormalization-group approach to remove the divergences and to make transparent the scaling symmetry of the problem.

III. RENORMALIZATION

We apply renormalization-group (RG) theory to make use of the scaling symmetry of the system in the asymptotic limit to extract the universal content and at the same time remove divergences that occur for the evaluation of the bare functions in this limit [39,44,45]. The theory given in terms of the initial bare variables is mapped to a renormalized theory. This is achieved by a controlled rearrangement of the series for the vertex functions. Several asymptotically equivalent procedures serve this purpose. Here we will use two somewhat complementary approaches: zero mass renormalization (see [39] for instance) with a successive ε expansion [46] and the fixed-dimension massive RG approach [40]. The first approach is performed directly for the critical point. Results for critical exponents at physically interesting dimensions $d=2$ and 3 are calculated in an $\varepsilon=4-d$ expansion [46–49]. The second approach renormalizes off the critical limit but calculates the critical exponents directly in space dimensions $d=2,3$ [50,51]. It also gives quantitative results for the preasymptotic critical behavior [52,53]. Most authors tend to prefer one method and to exclude the other for nonobvious reasons. The application of both approaches will enable us in particular to check the consistency of approximations and the accuracy of the results obtained.

Let us formulate the relations for a renormalized theory in terms of the corresponding renormalization conditions. Though they are different in principle for the two procedures, we may formulate them simultaneously using the same expressions. Note that the polymer limit of zero component fields leads to essential simplification. Each field ϕ_a, mass m_a, and coupling u_{aa} renormalizes as if the other fields were absent. First we introduce renormalized couplings g_{ab} by

$$u_{aa}=\mu^{\varepsilon}Z_{\phi_a}^{-2}Z_{aa}g_{aa}, \quad a=1,2 \tag{20}$$

$$u_{12}=\mu^{\varepsilon}Z_{\phi_1}^{-1}Z_{\phi_2}^{-1}Z_{12}g_{12}. \tag{21}$$

Here μ is a scale parameter equal to the renormalized mass at which the massive scheme is evaluated and sets the scale of the external momenta in the massless scheme. The renormalization factors Z_{ϕ_a},Z_{ab} are defined as power series in the renormalized couplings that fulfill the RG conditions

$$Z_{\phi_a}(g_{aa})\frac{\partial}{\partial k^2}\Gamma^{(2)}_{aa}(u_{aa}(g_{aa}))=1, \tag{22}$$

$$Z^2_{\phi_a}(g_{aa})\Gamma^{(4)}_{aaaa}(u_{aa}(g_{aa}))=\mu^{\varepsilon}g_{aa}, \tag{23}$$

$$Z_{\phi_1}(g_{11})Z_{\phi_2}(g_{22})\Gamma^{(4)}_{1122}(u_{ab}(g_{ab}))=\mu^{\varepsilon}g_{12}. \tag{24}$$

These formulas are applied perturbatively while the corresponding loop integrals are evaluated for zero external momenta in the massive approach and for external momenta at the scale of μ in the massless approach as explained in Appendix A. In the massive case the RG condition for the vertex function $\Gamma^{(2)}$ reads

$$Z_{\phi_a}(g_{aa})\Gamma^{(2)}_{aa}(u_{aa}(g_{aa}))\big|_{k^2=0}=\mu^2, \quad a=1,2. \tag{25}$$

In the case of massless renormalization the corresponding condition reads [39]:

$$Z_{\phi_a}(g_{aa})\Gamma^{(2)}_{aa}(u_{aa}(g_{aa}))\big|_{k^2=0}=0, \quad a=1,2. \tag{26}$$

In order to renormalize the star vertex functions we introduce renormalization factors Z_{*f_1,f_2} by

$$Z_{\phi_1}^{f_1/2}Z_{\phi_2}^{f_2/2}Z_{*f_1,f_2}\Gamma^{(*f_1 f_2)}(u_{ab}(g_{ab}))=\mu^{\delta_{f_1+f_2}}. \tag{27}$$

In the same way we define the appropriate renormalization for the vertex function of mutually avoiding walks (MAWs)

$$Z_{\phi_1}^{f/2} Z_{(\mathrm{MAW}f)} \Gamma^{*f}_{\mathrm{MAW}}(u_{12}(g_{ab})) = \mu^{\delta_f}. \tag{28}$$

The powers of μ absorb the engineering dimensions of the bare vertex functions. These are given by

$$\delta_f = f(\varepsilon/2 - 1) + 4 - \varepsilon. \tag{29}$$

The renormalized couplings g_{ab} defined by relations (20) and (21) depend on the scale parameter μ. The renormalization Z factors are power series in g_{ab} and thus implicitly depend on μ. This dependence is expressed by the RG functions defined by the relations

$$\mu \frac{d}{d\mu} g_{ab} = \beta_{ab}(g_{a'b'}), \tag{30}$$

$$\mu \frac{d}{d\mu} \ln Z_{\phi_a} = \eta_{\phi_a}(g_{aa}), \tag{31}$$

$$\mu \frac{d}{d\mu} \ln Z_{*f_1 f_2} = \eta_{*f_1 f_2}(g_{ab}), \tag{32}$$

$$\mu \frac{d}{d\mu} \ln Z_{\mathrm{MAW}f} = \eta_f^{\mathrm{MAW}}(g_{ab}). \tag{33}$$

The function η_{ϕ_a} describes the pair-correlation critical exponent, while the functions $\eta_{*f_1 f_2}$ and $\eta_f^{\mathrm{MAW}}(g_{ab})$ define the set of exponents for copolymer stars and stars of mutually avoiding walks. Note that Z_{*20} renormalizes the vertex function with a ϕ^2 insertion that coincides with $\Gamma^{(*20)}$. Consequently, the usually defined correlation length critical exponent ν is expressed in terms of functions η_{*20} and η_ϕ (see the next section). Explicit expressions for the β and η functions will be given in the next section together with a study of the RG flow and the fixed points of the theory.

IV. RENORMALIZATION-GROUP FLOW AND THE FIXED POINTS: ε EXPANSION AND PSEUDO-ε EXPANSION

Here we discuss the RG flow of the theory presented in Sec. III. In particular, we want to find appropriate representations for the fixed points of the flow. In a study devoted to ternary polymer solutions, the RG flow has been calculated [30] within massless renormalization and is known to the third-loop order of the ε expansion. Note that for the diagonal coupling g_{aa} the corresponding expressions are also found in the polymer limit $m=0$ of the $O(m)$ symmetric ϕ^4 model. They are known in even higher orders of perturbation theory [54]. To third-loop order the expressions read

$$\begin{aligned}
\beta^{\varepsilon}_{g_{aa}} &= -\varepsilon g_{aa} + \frac{1}{3}(4 + 2\varepsilon + 2\varepsilon^2) g_{aa}^2 \\
&\quad - \frac{1}{9}\left(\frac{21}{2} + \frac{215}{8}\varepsilon - 11 J\varepsilon \right) g_{aa}^3 \\
&\quad + \frac{1}{27}[79 - 22J + 33\zeta(3)] g_{aa}^4 + O(g_{aa}^5), \\
&\qquad a = 1,2
\end{aligned} \tag{34}$$

$$\begin{aligned}
\beta^{\varepsilon}_{g_{12}} &= -\varepsilon g_{12} + \frac{1}{3}\left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} \right)(g_{11} + g_{22}) g_{12} \\
&\quad + \frac{1}{3}(2 + \varepsilon + \varepsilon^2) g_{12}^2 - \frac{1}{9}\left(\frac{5}{4} + \frac{55}{16}\varepsilon - \frac{3}{2} J\varepsilon \right)(g_{11}^2 + g_{22}^2) \\
&\quad \times g_{12} - \frac{1}{9}\left(3 + \frac{15}{2}\varepsilon - 3J\varepsilon \right)(g_{11} + g_{22}) g_{12}^2 \\
&\quad - \frac{1}{9}(2 + 5\varepsilon - 2J\varepsilon) g_{12}^3 + \frac{1}{54}(15 - J)(g_{11}^3 + g_{22}^3) g_{12} \\
&\quad + \frac{1}{27}\left(\frac{27}{2} + 9\zeta(3) - 6J \right)(g_{11}^2 + g_{22}^2) g_{12}^2 + \frac{1}{27}(7 - 3J) \\
&\quad \times g_{11} g_{22} g_{12}^2 + \frac{1}{27}[12 + 6\zeta(3) - 2J](g_{11} + g_{22}) g_{12}^3 \\
&\quad + \frac{1}{27}[6 + 3\zeta(3) - 2J] g_{12}^4 + O(g^5).
\end{aligned} \tag{35}$$

Here the Riemann zeta function with $\zeta(3) \approx 1.202$ and the constant $J \approx 0.7494$ occur. We use an index ε at β^{ε} to distinguish the β functions obtained in massless renormalization with a successive ε expansion from β^m obtained in massive field theory.

Similarly, performing renormalization in the massive scheme, we obtain the corresponding functions β^m. We present them using convenient variables $v_{ab} = D_2^m g_{ab}$ and introduce new functions $\beta^m_{v_{ab}} = D_2^m \beta^m_{g_{ab}}$. Here D_2^m is the one-loop integral calculated within massive field theory (see appendix A). This procedure defines a convenient numerical scale for the massive β functions. The expressions for the functions $\beta^m_{v_{ab}}$ read

$$\begin{aligned}
\beta^m_{v_{aa}} &= -(4-d) v_{aa} \left\{ 1 - \frac{4 v_{aa}}{3} + \frac{2}{9}\left[22\left(i_1 - \frac{1}{2} \right) + 2 i_2 \right] v_{aa}^2 \right. \\
&\quad + \frac{2}{27}(-89 + 310 i_1 + 8 i_2 + 3 i_2 d - 12 i_3 - 180 i_4 - 87 i_5 \\
&\quad \left. - 21 i_6 - 33 i_7 - 12 i_8) v_{aa}^3 \right\} + O(v^5), \quad a = 1,2.
\end{aligned} \tag{36}$$

$$\begin{aligned}
\beta^m_{v_{12}} &= -(4-d) v_{12} \left[1 - \frac{1}{3}(v_{11} + v_{22} + 2 v_{12}) + \frac{1}{3}\left(-v_{11}^2 - v_{22}^2 \right. \right. \\
&\quad - 2 v_{12} v_{11} - \frac{4 v_{12}^2}{3} - 2 v_{12} v_{22} + \frac{2 v_{12}^2 i_2}{3} + \frac{8 v_{12}^2 i_1}{3} \\
&\quad \left. + 4 v_{12} v_{22} i_1 + 2 v_{11}^2 i_1 + \frac{2 v_{22}^2 i_2}{3} + 4 v_{12} v_{11} i_1 + 2 v_{22}^2 i_1 \right) \\
&\quad \left. + \sum_{j,k,l} b^{jkl} v_{11}^j v_{22}^k v_{12}^l \right] + O(v^5).
\end{aligned} \tag{37}$$

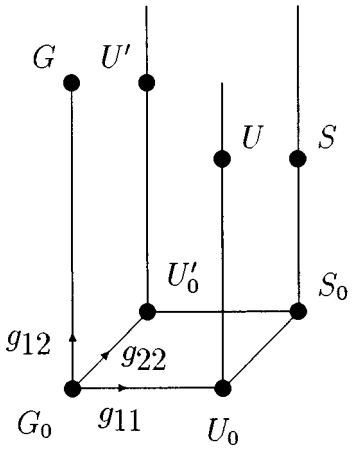

FIG. 4. Fixed points (FPs) of ternary polymer solution. The trivial FPs G_0, U_0, U_0', and S_0 correspond to the vanishing mutual interaction. The nontrivial FPs G, U, U', and S correspond to the nonvanishing mutual interaction ($g_{12}\neq 0$).

Here i_j are the dimension-dependent loop integrals, normalized by the one-loop integral value (see Appendix A). Expressions for the coefficients b^{jkl} are given in Appendix B. Note that the β function for the diagonal coupling g_{aa} is known within the massive scheme [50] to the order of six loops [55].

Let us solve the equations for the fixed points (FPs) $P(\{g_{11}^*,g_{22}^*,g_{12}^*\})$ of the β functions,

$$\beta^{\varepsilon}_{g_{aa}}(g_{aa}^*)=0, \quad a=1,2 \tag{38a}$$

$$\beta^{\varepsilon}_{g_{12}}(g_{11}^*,g_{22}^*,g_{12}^*)=0. \tag{38b}$$

As is well known, Eq. (38a) has two solutions $g_{aa}^*=0,g_S^*$. For Eq. (38b) one finds a total of eight FPs depending on the choice of g_{aa}^*. The trivial FPs are $G_0(0,0,0)$, $U_0(g_S^*,0,0)$, $U_0'(0,g_S^*,0)$, and $S_0(g_S^*,g_S^*,0)$, all corresponding to $g_{12}^*=0$. The nontrivial FPs are found as $G(0,0,g_G^*)$, $U(g_S^*,0,g_U^*)$, $U'(0,g_S^*,g_U^*)$, and $S(g_S^*,g_S^*,g_S^*)$. In the three-dimensional space of couplings g_{11},g_{22},g_{12} these FPs are placed at the corners of a cube deformed in the g_{12} direction (see Fig. 4). Their ε expansions read [30]

$$g_G^*=\frac{3\varepsilon}{2}-\left(J+\frac{3}{2}\zeta(3)\right)\frac{3\varepsilon^3}{8}, \tag{39}$$

$$g_U^*=\frac{9\varepsilon}{8}+\frac{39\varepsilon^2}{256}+\left(\frac{267}{4096}-\frac{693}{1024}\zeta(3)-\frac{189}{512}J\right)\varepsilon^3, \tag{40}$$

$$g_S^*=\frac{3\varepsilon}{4}+\frac{15\varepsilon^2}{128}+\left(\frac{111}{2048}-\frac{99}{256}\zeta(3)-\frac{33}{128}J\right)\varepsilon^3. \tag{41}$$

For the evaluation of the fixed points of the β functions calculated in the massive scheme (36) and (37) (as well as of the other quantities of the theory), one has several alternatives. The first possibility is to introduce ε expansions for the loop integrals. For massive renormalization these are known for the one- and two-loop integrals (see [45]):

$$D_2^m=\frac{1}{\varepsilon}\left(1-\frac{\varepsilon}{2}\right)+O(\varepsilon), \quad i_1=\frac{1}{2}+\frac{\varepsilon}{4}+O(\varepsilon^2),$$

$$i_2=-\frac{\varepsilon}{8}+O(\varepsilon^2). \tag{42}$$

A second possibility is to proceed directly in a fixed dimension of space $d=2$ or 3 substituting numerical values for the loop integrals [56,57]. For the massive two-loop ε expansion the fixed points read

$$g_S^*=\frac{3\varepsilon}{4}+\frac{111\varepsilon^2}{128}, \tag{43}$$

$$g_U^*=\frac{9\varepsilon}{8}+\frac{327\varepsilon^2}{256}, \tag{44}$$

$$g_G^*=\frac{3\varepsilon}{2}+\frac{3\varepsilon^2}{2}. \tag{45}$$

Only the first order of these results coincides with the fixed-point values of the massless renormalization scheme [30]. It is well known that the values of β functions, fixed points, and other intermediate functions in general depend on the RG scheme; only the critical exponents and other observables will be independent of the scheme followed.

We will now study expressions (36) and (37) directly at a fixed dimension. In this scheme the usual way of finding the fixed points of β functions of models with several couplings involves the numerical solution of the system of equations (38). To this end the asymptotic series in the coupling constants are represented in the form of corresponding resummed expressions $\beta^{\rm res}$ [58]. However, the numerical solution of the resummed fixed-point equation in general leads to inconsistent results, as we will show in Sec. VI. An alternative to this procedure and thus a third possibility to proceed was originally proposed by Nickel and may be called a pseudo-ε expansion [59]. To our knowledge, it has not, until now, been applied to theories with several couplings (see [51]), although it seems a convenient tool to circumvent the specific difficulties arising for the massive approach. To apply this method, we introduce the ''pseudo-ϵ'' parameter τ into the expressions for the β functions $\beta^m_{v_{aa}},\beta^m_{v_{12}}$ in Eqs. (36) and (37) as

$$-\beta^m_{v_{aa}}/(4-d)v_{aa}=\tau-\frac{4v_{aa}}{3}+\cdots, \quad a=1,2,$$

$$-\beta^m_{v_{12}}/(4-d)v_{12}=\tau-\frac{1}{3}(v_{11}+v_{22}+2v_{12})+\cdots. \tag{46}$$

We solve for the fixed-point solutions as series in τ. The resulting series for the fixed points then either can be resummed (to obtain the numerical values of the fixed points) or can be substituted into the expansions for the observables of the theory. In the final results we substitute $\tau=1$.

Performing this procedure, we get the fixed-point values as series in the pseudo-ϵ parameter τ up to the order τ^3:

$$v_G = 3/2\tau + (3\,i_1 - 3/2)\tau^2 + \left(-\frac{9i_6}{8} - \frac{9i_1}{4} + 3/8 + 12i_1^2 - \frac{9i_4}{2} - \frac{27i_5}{8} - \frac{9i_7}{8} \right)\tau^3, \tag{47}$$

$$v_U = \frac{9\tau}{8} + \left(\frac{93i_1}{32} - \frac{93}{64} + \frac{3i_2}{32} \right)\tau^2 - \left(\frac{387i_6}{512} + \frac{9i_8}{64} + \frac{33i_2}{64} + \frac{1281i_1}{256} - \frac{459}{512} + \frac{693i_7}{512} + \frac{27i_3}{128} + \frac{369i_4}{64} + \frac{1485i_5}{512} - \frac{3i_2^2}{64} - \frac{9i_2 i_1}{8} - \frac{969i_1^2}{64} - \frac{27di_2}{512} \right)\tau^3, \tag{48}$$

$$v_S = 3/4\tau + \left(\frac{3\,i_2}{16} + \frac{33\,i_1}{16} - \frac{33}{32} \right)\tau^2 + \left(3/4 - \frac{27i_2}{32} - \frac{261i_5}{128} - \frac{135i_4}{32} - \frac{261i_1}{64} - \frac{63i_6}{128} + \frac{3i_2^2}{32} + \frac{33i_2 i_1}{16} + \frac{9di_2}{128} - \frac{99i_7}{128} - \frac{9i_8}{32} - \frac{9i_3}{32} + \frac{363{i_1}^2}{32} \right)\tau^3. \tag{49}$$

Expressions (39)–(41) [30] and (36)–(49) give the fixed-point values of ternary solutions in the massless and massive renormalization schemes and are the main results to be used in the subsequent calculations.

Looking for the stability of the above-described fixed points, one finds that only the fixed point S is stable [30]. In the excluded-volume limit of infinitely long chains, the behavior of a system of two polymer species is thus described by the same scaling laws as a solution of only one polymer species. Nevertheless, taking into account that real polymer chains are not infinitely long, one may also find crossover phenomena that are governed by the unstable fixed points. Knowing the complete RG flow allows one to describe crossover phenomena in the whole accessible region [30]. However, for the purpose of our study we are interested only in the values of the fixed points and the properties of the star vertex functions at these fixed points.

V. RESULTS FOR EXPONENTS

For homogeneous stars of polymer chains of one species alone, several sets of star exponents have been defined, each describing either the scaling properties of the configurational number [see formula (3) of this article] or the anomalous dimensions of star vertices, etc. Due to scaling relations, these exponents can be expressed in terms of each other [24]. In this sense, each set of star exponents forms a complete basis. For the copolymer and MAW stars, we here choose to present our results in terms of the exponents $\eta_{f_1 f_2}$ and η_f^{MAW} given by the fixed-point values of the functions $\eta_{*f_1 f_2}(g_{ab})$ (32) and $\eta_f^{\mathrm{MAW}}(g_{ab})$ (33). Let us define the asymptotic values of copolymer star exponents and MAW star exponents by

$$\eta^S_{f_1 f_2} = \eta_{*f_1 f_2}(g_{ab})|_S, \tag{50}$$

$$\eta^G_{f_1 f_2} = \eta_{*f_1 f_2}(g_{ab})|_G, \tag{51}$$

$$\eta^U_{f_1 f_2} = \eta_{*f_1 f_2}(g_{ab})|_U = \eta_{*f_2 f_1}(g_{ab})|_{U'}, \tag{52}$$

$$\eta_f^{\mathrm{MAW}} = \eta_f^{\mathrm{MAW}}(g_{ab})|_G. \tag{53}$$

The exponent in the symmetrical FP S can also be expressed by $\eta^S_{f_1 f_2} = \eta^U_{f_1+f_2,0}$. Starting from the expressions for the fixed points given in Sec. IV and relations (50)–(53), we find the series for the star exponents. In the ε expansion we obtain the following expansions for $\eta_{f_1 f_2}$:

$$\eta^G_{f_1 f_2}(\varepsilon) = -f_1 f_2 \frac{\varepsilon}{2} + f_1 f_2 (f_2 - 3 + f_1)\frac{\varepsilon^2}{8} - f_1 f_2 (f_2 - 3 + f_1)[f_1 + f_2 + 3\,\zeta(3) - 3]\frac{\varepsilon^3}{16}, \tag{54}$$

$$\eta^U_{f_1 f_2}(\varepsilon) = f_1(1 - f_1 - 3f_2)\frac{\varepsilon}{8} + f_1(25 - 33f_1 + 8f_1^2 - 91f_2 + 42f_1 f_2 + 18f_2^2)\frac{\varepsilon^2}{256} + f_1[577 - 969f_1 + 456f_1^2 - 64f_1^3 - 2463f_2 + 2290f_1 f_2 - 492f_1^2 f_2 + 1050f_2^2 - 504f_1 f_2^2 - 108f_2^3 - 712\zeta(3) + 936f_1\zeta(3) - 224f_1^2\zeta(3) + 2652f_2\zeta(3) - 1188f_1 f_2\zeta(3) - 540f_2^2\zeta(3)]\frac{\varepsilon^3}{4096}, \tag{55}$$

$$\eta^{\mathrm{MAW}}_{f_1 f_2}(\varepsilon) = -(f_1 - 1)f_1\frac{\varepsilon}{4} + f_1(f_1 - 1)(2f_1 - 5)\frac{\varepsilon^2}{16} - (f_1 - 1)f_1[4f_1^2 - 20f_1 + 8f_1\zeta(3) - 19\zeta(3) + 25]\frac{\varepsilon^3}{32}. \tag{56}$$

Here $\zeta(3) \simeq 1.202$ is the Riemann zeta function. The above formulas reproduce the third-order calculations of the scaling exponents of homogeneous polymer stars $\gamma_f - 1 = \nu(\eta^U_{f,0} - f\,\eta^U_{2,0})$ [32]. The exponents $\lambda^{(xx)}$ given to second order in Eqs. (xx) of [5] to describe the multifractal scaling properties of a Laplacian field with fractal boundary conditions are reproduced following $\lambda^{(29)}(n) = -\eta^G_{2,n}$, $\lambda^{(47)}(n) = -\eta^U_{2,n} + \eta^U_{2,0}$, $\lambda_e^{(48)}(n) = -\eta^G_{1,n}$, and $\lambda_e^{(49)}(n) = -\eta^U_{1,n}$, correcting a misprint in Eq. (49) of [5]. Also the second-order results for exponents $x_{L,n} - x_{L,1} = -2(\eta^G_{L,n} - \eta^G_{L,1})$ of [1] and the MAW exponents $\sigma_L = 1/2\,\eta_L^{\mathrm{MAW}}$ defined in [11] find their third-order extension by the above expansions. The pseudo-ε expansions for $\eta_{f_1 f_2}$ obtained in the massive scheme read

$$\eta^G_{f_1 f_2} = \tau\epsilon\frac{f_1 f_2}{2}\Big[-1 + (f_2 - 3 + f_1)\Big(\tau\,(i_1 - 1/2) + \frac{\tau^2}{8}$$

TABLE I. Values of the copolymer star exponent $\eta_{f_1 f_2}$ obtained in the first, second, and third order in the Gaussian (G) fixed point in the ε expansion and pseudo-ε (τ) expansion for different values of f_1, f_2 at $\varepsilon = 1$ ($d = 3$). R stands for the results obtained by Padé-Borel resummation of the three-loop series.

f_1	f_2	$\sim\varepsilon$	$\sim\varepsilon^2$	$\sim\varepsilon^3$	R	$\sim\tau$	$\sim\tau^2$	$\sim\tau^3$	R
1	1	−0.50	−0.63	−0.46	−0.56	−0.50	−0.58	−0.56	−0.57
1	2	−1.00	−1.00	−1.00		−1.00	−1.00	−1.00	
1	3	−1.50	−1.13	−1.99	−1.36	−1.50	−1.25	−1.42	−1.34
2	2	−2.00	−1.50	−2.65	−1.81	−2.00	−1.67	−1.93	−1.80
2	3	−3.00	−1.50	−5.71	−2.50	−3.00	−2.00	−3.01	−2.45
3	3	−4.50	−1.13	−12.27	−3.48	−4.50	−2.25	−5.09	−3.37

$$\times[32\,(i_1-1/2)^2+6i_6-18i_4-8+22i_1-6i_7-6i_5 -(f_2-3+f_1)(2+6i_4-6i_1)]\Big)-\frac{\tau^2}{4}(f_1f_2-2) \times(1+3i_5+3i_6-6i_1)\Big], \tag{57}$$

$$\eta^U_{f_1f_2}=\tau\epsilon\,\frac{f_1}{1024}\Big(128-128f_1-384f_2+\tau[(288i_1-144)f_2^2 -208+416i_1+32i_2+(272-32i_2-544i_1)f_1 +(-32i_2-1472i_1+736)f_2+(-64+128i_1)f_1^2 +(-336+672i_1)f_1f_2]+\tau^2\sum_{k_1,k_2} f_1^{k_1}f_2^{k_2}\eta_{U;k_1;k_2}\Big), \tag{58}$$

$$\eta^{\mathrm{MAW}}_f=\tau\,\epsilon\,\frac{f(f-1)}{16}\{-4-(20i_1-10+4f-8fi_1)\tau +[(-3i_5+18i_1-12i_4-3i_6-5)f^2+(3i_5+32i_1^2 -8i_7+15i_6-88i_1+42i_4+23)f-26-18i_6+19i_7 +12i_5+106i_1-80i_1^2-30i_4]\tau^2\}. \tag{59}$$

The expressions for the three-loop terms $\eta_{U;k_1;k_2}$ in Eq. (58) are given in Appendix B. It has been pointed out in [5] that for the exponent $\eta^G_{12}=-\lambda^{(29)}(1)$ (see above) an exact estimate equal to our first-order contribution may be found. It is indeed remarkable that all higher-order contributions to η^G_{12} appear to vanish in both approaches.

With these exponents we can describe the scaling behavior of polymer stars and networks of two components, generalizing the relation for single-component networks [24]. In the notation of Eq. (4) we find for the number of configurations of a network $\mathcal{G}$ of F_1 and F_2 chains of species 1 and 2,

$$\mathcal{Z}_{\mathcal{G}}\sim(R/\ell)^{\eta_{\mathcal{G}}-F_1\eta_{20}-F_2\eta_{02}},$$

$$\text{with}\quad \eta_{\mathcal{G}}=-dL+\sum_{f_1+f_2\geq 1} N_{f_1f_2}\eta_{f_1f_2}, \tag{60}$$

where L is the number of loops and $N_{f_1f_2}$ the number of vertices with f_1 and f_2 arms of species 1 and 2 in the network $\mathcal{G}$. To receive an appropriate scaling law we assume the network to be built of chains that for both species will have a coil radius R when isolated.

For the sake of completeness we give also the results in the ε and pseudo-ε expansions for the correlation length critical exponent $\nu=1/(2+\eta_{20}-\eta_\phi)$ and the pair-correlation function critical exponent η_ϕ in the nontrivial fixed point:

$$\eta_\phi(\varepsilon)=\frac{\varepsilon^2}{64}+\frac{17\varepsilon^3}{1024}, \tag{61}$$

$$\nu(\varepsilon)=\frac{1}{2}+\frac{1}{16}\varepsilon+\frac{15}{512}\varepsilon^2+\left(\frac{135}{8192}-\frac{33\zeta(3)}{1024}\right)\varepsilon^3, \tag{62}$$

$$\eta_\phi(\tau)=\frac{-(4-d)}{128}\tau(16\tau i_2-12\tau^2 i_2-24\tau^2 i_8+8\tau^2 i_2^2 +88\tau^2 i_1 i_2), \tag{63}$$

$$\nu(\tau)=\frac{1}{2}+\frac{4-d}{16}\tau-\frac{4-d}{512}(4-40i_1+8i_2+4d)\tau^2 -\frac{4-d}{512}\Big(6+12i_2+10i_1+84i_4+63i_5+3i_6+33i_7 -12i_8-220i_1^2+4i_2^2+24i_1i_2-d+10i_1d -2i_2d-\frac{d^2}{2}\Big)\tau^3. \tag{64}$$

VI. RESUMMATION

As is well known, the perturbation series expansions of renormalized field theory are nonconvergent, but generally assumed to be asymptotic. For the exponents $\eta_{f_1f_2}$ this behavior is indicated in the corresponding columns of Tables I and II, where the series are summed without further analysis.

The increase in the coefficients of the high-order terms of perturbation theory series may be estimated using information such as the combinatorial growth of the contributions with order. The series for the β function of the $O(m)$ symmetric ϕ^4 model with one coupling g has the asymptotic behavior [60,61]

$$\beta(g)=\sum_k A_k g^k, \tag{65}$$

TABLE II. Values of the copolymer star exponent $\eta_{f_1 f_2}$ obtained in the first, second, and third order in the unsymmetrical (U) fixed point in the ε expansion and pseudo-ε (τ) expansion for different values of f_1, f_2 at $\varepsilon = 1$ ($d = 3$). R stands for the results obtained by Padé-Borel resummation of the three-loop series.

f_1	f_2	$\sim\varepsilon$	$\sim\varepsilon^2$	$\sim\varepsilon^3$	R	$\sim\tau$	$\sim\tau^2$	$\sim\tau^3$	R
1	1	−0.38	−0.50	−0.28	−0.43	−0.38	−0.46	−0.43	−0.44
1	2	−0.75	−0.85	−0.69	−0.80	−0.75	−0.82	−0.78	−0.80
1	3	−1.13	−1.07	−1.33	−1.11	−1.13	−1.09	−1.11	−1.10
2	1	−1.00	−0.98	−0.71	−1.00	−1.00	−0.99	−0.98	−0.99
2	2	−1.75	−1.37	−2.37	−1.62	−1.75	−1.50	−1.71	−1.60
2	3	−2.50	−1.47	−4.99	−2.19	−2.50	−1.82	−2.56	−2.13
3	1	−1.88	−1.28	−1.70	−1.50	−1.88	−1.48	−1.82	−1.64
3	2	−3.00	−1.36	−6.19	−2.47	−3.00	−1.91	−3.18	−2.43
3	3	−4.13	−1.02	−12.83	−3.26	−4.13	−2.06	−4.97	−3.14

$$A_k = c k^{b_0} (-a)^k k! [1 + O(1/k)], \quad k \to \infty. \tag{66}$$

The quantities a, b_0, c were calculated in [60,62]. A similar behavior is found for the critical exponents expressed as a series in powers of the coupling. The same results also apply to the divergence of the ε and pseudo-ε expansions derived above. The property (66) indicates the Borel summability of the series $\beta(g)$ [63]. The Borel resummation procedure takes into account the asymptotic behavior of the coefficients and maps the asymptotic series to a convergent series with the same asymptotic limit. The function β_{aa} (30) coincides with the $O(m)$ symmetric β function (65) in the polymer limit $m = 0$. So its asymptotic behavior is known. The asymptotic behavior of the off-diagonal β function β_{12} was found by instanton analysis (see [64,65]) in [30].

Let us introduce the techniques for resummation, using the known asymptotic behavior of the series. Here we apply the Padé-Borel resummation [50] and a resummation refined by a conformal mapping [66]. The first way of resummation is applicable only for alternating series, while the second one is more universal. The resummation procedures are as follows [50,51,63,66]. For an asymptotic series

$$f(\varepsilon) = \sum_j f^{(j)} \varepsilon^j, \tag{67}$$

one defines the Borel-Leroy transform $f^B(\varepsilon)$ of the series by

$$f^B(\varepsilon) = \sum_j \frac{f^{(j)} \varepsilon^j}{\Gamma(j + b + 1)}, \tag{68}$$

with the Euler Gamma function (b is a fit parameter). Then the initial series may be regained from

$$f^{\mathrm{res}}(\varepsilon) = \int_0^\infty dt\ t^b e^{-t} f^B(\varepsilon t). \tag{69}$$

Substituting for $f^B(\varepsilon)$ its analytic continuation in the form of a Padé approximant and evaluating Eq. (69) for the truncated series, this procedure constitutes the Padé-Borel resummation [50,63]. The conformal mapping technique in addition uses the constant a in Eq. (66). Assuming that the behavior (66) holds also for the expansion of $f(\varepsilon)$ in ε, one concludes that the singularity of the transformed series $f^B(\varepsilon)$ closest to the origin is located at the point $(-1/a)$. Conformally mapping the ε plane onto a disk of radius 1 while leaving the origin invariant,

$$w = \frac{(1 + a\varepsilon)^{1/2} - 1}{(1 + a\varepsilon)^{1/2} + 1}, \quad \varepsilon = \frac{4}{a} \frac{w}{(1 - w)^2},$$

substituting this into $f^B(\varepsilon)$, and expanding in w, we receive a series defined on the disk with radius 1 on the w plane. This series is then resubstituted into Eq. (69). In order to weaken a possible singularity in the w plane the corresponding expression is multiplied by $(1 - w)^\alpha$ introducing an additional parameter α [66]. In the resummation procedure the value of a is taken from the known high-order behavior of the ε-expansion series, while α is chosen in our calculations as a fit parameter defined by the condition of minimal difference between resummed second-order and third-order results. The resummation procedure was seen to be quite insensitive to the parameter b introduced in the Borel-Leroy transformation (68) [51].

For the resummation of the exponents $\eta_{f_1 f_2}$ we take into account the combinatorial factors that multiply each contribution according to the numbers of chains f_1 and f_2. We include an additional factor $(f_1 + f_2)^k$ for the kth-order contributions, multiplying the constant a by $f_1 + f_2$. For resummation of the series at the fixed points S, G, and U, the following values of $a = a^S, a^G, a^U$ are used [60,30]:

$$a^S = a^G = 3/8, \quad a^U = 27/64. \tag{70}$$

By analogy we use the same procedures developed for the ε expansion also for the τ expansion, which we assume to have the same asymptotic behavior as it is in the same way collecting contributions of the same loop order.

Let us note here that the conventional resummation of the β functions in the massive approach leads to a severe inconsistency, which is the reason for us to take the pseudo-ε- or τ-expansion method. The distinct feature of the β_{ab} functions introduced here is that they are functions of different numbers of variables, which leads to ambiguities in their analytical continuation via Padé approximants or rational approximants of several variables (see [67]). Let us illustrate this for the example of the two-loop approximation. The corresponding expressions read

TABLE III. Values of the copolymer star exponent $\eta^G_{f_1 f_2}$ at $d=3$ obtained by the ε expansion (ε) and the fixed-dimension technique ($3d$).

f_2 \ f_1	1		2		3		4		5		6	
	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$
1	−0.56	−0.58	−1.00	−1.00	−1.33	−1.35	−1.63	−1.69	−1.88	−1.98	−2.10	−2.24
2			−1.77	−1.81	−2.45	−2.53	−3.01	−3.17	−3.51	−3.75	−3.95	−4.28
3					−3.38	−3.57	−4.21	−4.50	−4.94	−5.36	−5.62	−6.15
4							−5.27	−5.71	−6.24	−6.84	−7.12	−7.90
5									−7.42	−8.24	−8.50	−9.54
6											−9.78	−11.07

$$\beta_{v_{aa}} = -(4-d)v_{aa} f_{v_{aa}}(v_{aa}), \quad a=1,2 \tag{71}$$

$$\beta_{v_{12}} = -(4-d)v_{12} f_{v_{12}}(v_{11}, v_{22}, v_{12}), \tag{72}$$

with obvious expressions for $f_{v_{11}}, f_{v_{22}}, f_{v_{12}}$. In order to obtain the analytical continuation of the Borel transformed functions of one variable $f_{v_{aa}}(v_{aa})$ (71) one can make use of the [1/1] Padé approximant. Solving the corresponding nonlinear equations numerically, we find for the nontrivial fixed point S, $v_{11}=v_{22}=1.1857$ [68]. In order to apply a similar resummation technique to the function $f_{v_{12}}(v_{11}, v_{22}, v_{12})$ (72) one can make use of a generalization of Padé approximants to the case of several variables, i.e., represent the Borel transform $f^B_{v_{12}}$ of $f_{v_{12}}$ in the form of a rational approximant f^P of three variables [67]

$$f^P_{v_{12}}(v_{11}t, v_{22}t, v_{12}t) = \frac{1 + a_1(v_{11}, v_{22}, v_{12})t + a_2(v_{11}, v_{22}, v_{12})t^2}{1 + b(v_{11}, v_{22}, v_{12})t}. \tag{73}$$

In spite of the fact that the rational approximant (73) preserves the projection properties of the initial series (72), i.e., setting any pair of variables $\{v_{11}, v_{22}, v_{12}\}$ equal to zero in Eq. (73) one gets the appropriate [1/1] Padé approximant for the remaining variable, the ''global'' symmetry is not preserved. Due to different analytical continuations for the Borel transforms of the series (71) on the one hand and the series (72) on the other, solving the fixed-point equation for the resummed function

$$f^{\mathrm{res}}_{v_{12}} = 0, \tag{74}$$

we will never obtain a symmetrical solution $v^*_{11}=v^*_{22}=v^*_{12} \neq 0$. For the fixed point S we substitute $v_{11}=v_{22}=1.1857$ while solving Eq. (74) we obtain $v_{12}=0.9571$ [70]. The reason is that substituting *numerical* values of fixed-point coordinates v_{11}, v_{22} into Eq. (74) we lose information about the contributions to the fixed-point value from different orders of the perturbation theory series. So it appears quite natural to restore this information by generalizing the pseudo-ε expansion [59] to the case of several couplings as described in Sec. IV.

VII. NUMERICAL RESULTS

In the following we present our numerical results for the exponents $\eta^G_{f_1 f_2}$, $\eta^U_{f_1 f_2}$, and η_f^{MAW}. The exponent in the symmetrical fixed point S is included due to the relation $\eta^S_{f_1 f_2} = \eta^U_{f_1+f_2,0}$ [71]. Numerical results for the exponent $\gamma_f - 1 = \nu(\eta^U_{f0} - f\,\eta^U_{20})$ may be found in the ε expansion in [32] and in the pseudo-ε expansion in [72].

A. $d=3$

Let us first consider the case $d=3$. Tables I and II show some of the resummed results for the ε and τ expansions in comparison with the naive resummation of the series. While the nonresummed results differ to a great extent for the two approaches at high f_1, f_2, resummation shows that the two schemes yield consistent numerical estimates. Tables III–V list our final results using the resummation procedure refined by the conformal mapping technique as described in Sec. VI.

TABLE IV. Values of the copolymer star exponent $\eta^U_{f_1 f_2}$ at $d=3$ obtained by the ε expansion (ε) and the fixed-dimension technique ($3d$).

f_2 \ f_1	1		2		3		4		5		6	
	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$
0	0	0	−0.28	−0.28	−0.75	−0.76	−1.36	−1.38	−2.07	−2.14	−2.88	−3.01
1	−0.43	−0.45	−0.98	−0.98	−1.64	−1.67	−2.39	−2.47	−3.21	−3.38	−4.11	−4.40
2	−0.79	−0.81	−1.58	−1.60	−2.44	−2.52	−3.33	−3.50	−4.28	−4.57	−5.29	−5.73
3	−1.09	−1.09	−2.13	−2.19	−3.16	−3.30	−4.20	−4.48	−5.28	−5.71	−6.41	−7.03
4	−1.35	−1.37	−2.61	−2.71	−3.82	−4.04	−5.02	−5.40	−6.24	−6.81	−7.48	−8.28
5	−1.60	−1.64	−3.05	−3.21	−4.44	−4.75	−5.80	−6.30	−7.15	−7.89	−8.51	−9.50
6	−1.81	−1.89	−3.46	−3.68	−5.01	−5.42	−6.53	−7.15	−8.02	−8.92	−9.50	−10.69

TABLE V. Values of η_f^{MAW} exponents of star of mutually avoiding walks at $d=3$ and 2 obtained by the ε expansion (ε) and the fixed-dimension technique ($3d$,$2d$). The last column gives the exact conjecture at $d=2$ [11,12].

	$d=3$		$d=2$		
f	ε	$3d$	ε	$2d$	exact
1	0	0	0	0	$-.250$
2	$-.56$	$-.56$	-1.20	-1.19	-1.250
3	-1.38	-1.36	-2.71	-2.60	$-2.916(6)$
4	-2.36	-2.34	-4.36	-4.07	-5.250
5	-3.43	-3.43	-6.04	-5.61	-8.250
6	-4.58	-4.64	-7.78	-7.17	$-11.916(6)$

Comparing the numerical values listed in the above tables, it is convincing that the two approaches and the different resummation procedures all lead to results that lie within a bandwidth of consistency, which is broadening for larger values of number of chains $f_1, f_2 > 1$. This is not surprising as we have seen in Sec. V that our expansion parameters are multiplied by f_1 and f_2. Rather it is remarkable that even for a total number of chains of the order of 10 (see Tables III and IV) we still obtain results that are comparable to each other.

It seems noteworthy that at least for low numbers of chains ($f_1+f_2 \sim 4$) the nonresummed τ expansion seems to give results that do not differ essentially from the resummed values. Also the nonrefined Padé-Borel results of the τ expansion are closer to the refined summation of the ε expansion.

Does the data answer the question of convexity of the spectrum? A close study of the matrix of values reveals, that for fixed f_1 both $\eta^G_{f_1 f_2}$ and $\eta^U_{f_1 f_2}$ are convex from above as functions of f_2, thus yielding ''multifractal (MF) statistics.'' The relation to a MF spectral function for $f_1=1,2$ has been pointed out in [5]; it is analyzed in close detail in view of the present data and field theoretic formulation elsewhere [38]. On the other hand, also copolymer stars should repel each other. This is found to be true as well; the corresponding convexity from below shows up, e.g., along the diagonal values η_{ff} as a function of f. The general relation $\eta_{f_1 f_2} + \eta_{f_1' f_2'} \geqslant \eta_{f_1+f_1', f_2+f_2'}$ is always fulfilled. Thus, even though simple power- (k) of-field operators ϕ^k do not describe MF moments [1], they may be written as a power-($L+k$) of-field operators of suitable symmetry that have the appropriate short-distance behavior. This is also illustrated in the next subsection by Fig. 5, showing the spectrum of exponents $\eta^{\mathrm{G}}_{f_1 f_2}$ in the 2D limit. The opposite convexity along the diagonal as opposed to each of the two axes is clearly seen for these combinations of two random-walk stars that mutually interact.

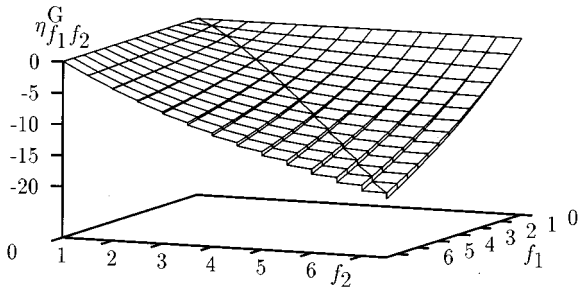

FIG. 5. Exponent $\eta^G_{f_1 f_2}$ in the ''Gaussian'' fixed point at $d=2$ obtained in the ε expansion and in the fixed d scheme. Steps on the ''flying carpet'' correspond to the difference of the results of the two renormalization-group approaches. The diagonal line shows the values η^G_{ff}.

B. $d=2$

While two-dimensional star polymers up to now have not found an experimental realization, their study is of some theoretical interest. It has been shown that the scaling dimensions of two-dimensional uniform polymer stars belong to a limiting case of the so-called conformal Kac table [73–75]. They have been calculated exactly by Coulomb gas techniques [17,35]. An exact relation has also been proposed for stars of mutually avoiding walks [11,12]. But it is still an open question if exact results for the copolymer star system may be derived in this formalism. Our numerical results for the exponents $\eta^G_{f_1 f_2}$, $\eta^U_{f_1 f_2}$, and $\eta^{\mathrm{MAW}}_{f_1 f_2}$ are presented in Tables VI, VII, and V, respectively.

Exact results for exponents of two-dimensional systems that are described by a conformal field theory with central charge $c<1$ may be taken from the Kac table of scaling dimensions [73–75]

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{75}$$

where p,q are integers in the minimal block

TABLE VI. Values of the copolymer star exponent $\eta^G_{f_1 f_2}$ at $d=2$ obtained by the ε expansion (ε) and the fixed-dimension technique ($2d$).

f_2 \ f_1	1		2		3		4		5		6	
	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$
1	-1.20	-1.22	-1.98	-1.98	-2.56	-2.58	-2.99	-3.04	-3.36	-3.43	-3.68	-3.78
2			-3.41	-3.45	-4.49	-4.59	-5.37	-5.52	-6.13	-6.34	-6.80	-7.04
3					-6.05	-6.23	-7.36	-7.63	-8.49	-8.84	-9.50	-9.91
4							-9.06	-9.44	-10.55	-11.03	-11.89	-12.45
5									-12.38	-12.98	-14.03	-14.74
6											-15.99	-16.81

TABLE VII. Values of the copolymer star exponent $\eta^U_{f_1 f_2}$ at $d=2$ obtained by the ε expansion (ε) and the fixed-dimension technique ($2d$).

f_2 \ f_1	1		2		3		4		5		6	
	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$	ε	$3d$
0	0	0	−0.59	−0.62	−1.51	−1.53	−2.61	−2.63	−3.84	−3.89	−5.18	−5.28
1	−0.91	−0.96	−1.94	−1.96	−3.11	−3.13	−4.35	−4.41	−5.71	−5.83	−7.17	−7.35
2	−1.62	−1.63	−3.05	−3.09	−4.49	−4.54	−5.94	−6.06	−7.46	−7.64	−9.04	−9.30
3	−2.16	−2.16	−4.00	−4.04	−5.70	−5.80	−7.39	−7.57	−9.09	−9.35	−10.82	−11.17
4	−2.60	−2.64	−4.80	−4.88	−6.79	−6.96	−8.72	−8.97	−10.61	−10.95	−12.52	−12.95
5	−3.00	−3.03	−5.52	−5.63	−7.81	−8.01	−9.96	−10.27	−12.06	−12.47	−14.14	−14.65
6	−3.34	−3.39	−6.17	−6.33	−8.73	−8.99	−11.12	−11.49	−13.43	−13.91	−15.69	−16.27

$$1 \leqslant p \leqslant m-1, \quad 1 \leqslant q \leqslant p, \tag{76}$$

and m is connected with the central charge c by

$$c = 1 - 6/m(m+1), \quad m \geqslant 3. \tag{77}$$

The exact result for the star exponents of uniform stars in two dimensions is obtained in the sublimiting case of $m=2$ (which means $c=0$) for half integer values of p [17,35],

$$x_f = 2h_{f/2,0} = (9f^2-4)/48. \tag{78}$$

The scaling dimension x_f is related to the exponent η_f by

$$x_f = \frac{1}{2} f(d-2+\eta) - \eta_f . \tag{79}$$

For the exponents of the star of MAW the following result was conjectured for $d=2$ [11,12,14]:

$$\eta_f^{\mathrm{MAW}} = x_f^{\mathrm{MAW}} = 2h_{0,f} = \frac{1-4f^2}{12}. \tag{80}$$

These values are shown in the last column of Table V. Plotting the resummed data for η_f^{MAW} from Table V with respect to f^2, one finds good agreement with the conjectured slope of $-1/3$.

The qualitative behavior of the exponent $\eta^G_{f_1 f_2}$ in the Gaussian fixed point is shown in Fig. 5. The steps in the ''flying carpet'' correspond to the difference of the results of the two RG approaches. Note that the curvature of the surface along the diagonal in the f_1-f_2 plane has opposite sign to that along each of the axes. From this curvature it is obvious that the dependence of the exponent on f_1, f_2 may not be described by a simple second-order polynomial. The best fit we could find to our resummed data using a simple formula that reproduces the vanishing result for $f_1+f_2=3$ found in the ε expansion reads

$$\eta^{G,\mathrm{app}}_{f_1 f_2} = -f_1 f_2 [a + b/(f_1+f_2)], \quad a = 1/4, \quad b = 3(1-a). \tag{81}$$

Note that the right-hand side of Eq. (81) vanishes if f_1 or f_2 is zero according to our perturbative results. This might be a defect of the perturbation theory as a finite result may be expected in $d=2$ as in Eqs. (78) and (80) evaluated for $f=0$.

In two dimensions however, each chain of a star will interact only with its direct neighbors. A star described here by η^{G}_{ff} will behave like a MAW $2f$ star if each species-1 chain has two neighbors of species 2, whereas it will behave differently if the chains are ordered such that each species is in one bulk of chains. The 2D copolymer stars in this sense reveal an even richer behavior. Thus the copolymer generalization of the MAW star adds another problem, for which a rigorous formulation in terms of an exactly solvable 2D model is yet to be found.

VIII. CONCLUSIONS AND OUTLOOK

Several reasons motivated our study. First, we intended to reveal the scaling behavior of copolymer stars and networks in solutions generalizing former studies of homogeneous polymer networks. This included revisiting the theory of ternary polymer solutions and adding an independent approach to the calculations. Second, the description of multifractal spectra in terms of random walks [5] promised to prove the relation of field theory and multifractals for this case. In particular we intended to check the convexity properties expected for the spectrum of exponents. A third motivation arose from the known peculiarities of polymers and polymer stars in two dimensions. Apart from numerically verifying previous results on polymer and mutually avoiding walk stars, we pose the question of finding an exactly solvable (conformal) two-dimensional theory for general copolymer stars.

We have extensively studied the spectrum of exponents governing the scaling properties of stars of walks taking into account the self- and mutual interactions of a system of species of polymers. Our study was performed in the framework of field-theoretical RG theory using two complementary approaches: the renormalization at zero mass in conjunction with the ε expansion and massive renormalization at fixed dimension with numerical evaluation of loop integrals. We have formulated the problem of finding the scaling exponents of stars of interacting and noninteracting walks in terms of the determination of the scaling dimensions of composite field operators of Lagrangian field theory. On the one hand, this allows for the application of well-developed formalisms and methods for analyzing the scaling properties. On the other hand, this defines these families of exponents extending previous sets in the framework of Lagrangian field theory. Our results agree with the previous studies of special

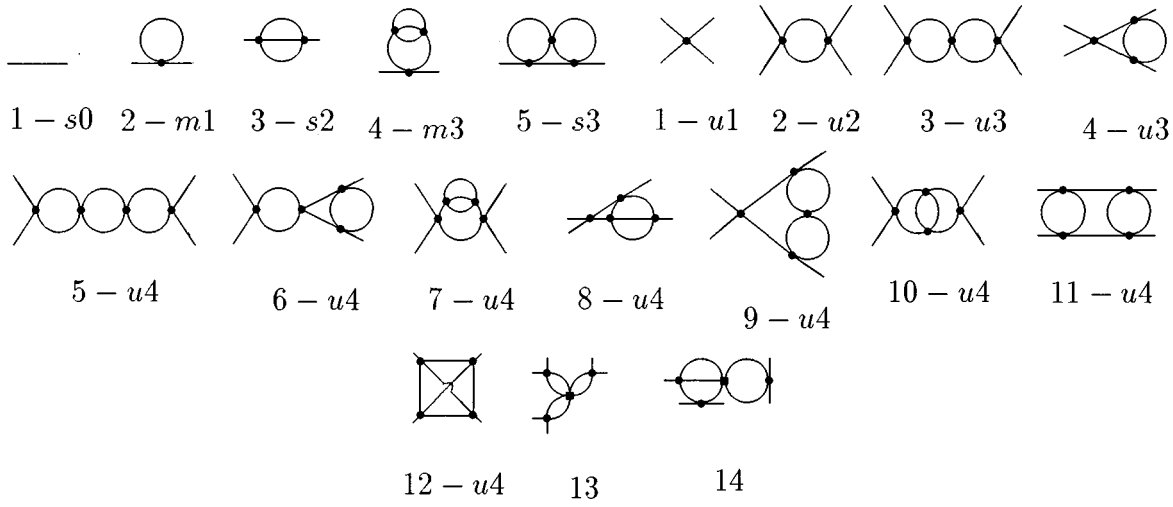

FIG. 6. Graphs of functions $\Gamma^{(2)}$, $\Gamma^{(4)}$ in the three-loop approximation. Graphs 13 and 14 represent additional contributions to the function $\Gamma^{(*f)}$. In graphs 13 and 14 the f vertex is indicated by a box.

cases that were in part done only to second order of the ε expansion. We have here considered the general case of a star of two mutually avoiding sets of walks, the walks of each set either self-interacting or not. Also we have studied the case of a star of mutually interacting walks. All calculations were performed to third order of perturbation theory. The sets of exponents are given in the ε expansion [formulas (54)–(56)] and in terms of the pseudo-ε expansion [formulas (57)–(59)]. The latter has proved to be a most suitable tool to evaluate this massive theory containing serveral couplings. We have shown that the conventional way of direct solution even of the resummed expressions for the fixed points of the theory would lead to severe problems in this case. We have evaluated the series obtained in both approaches for space dimensions $d=2$ and 3. Numerical values are produced by careful resummation of the asymptotic series using the results of an instanton analysis of the three-coupling problem [30]. For comparison we have also given the results of naive summation as well as standard Padé-Borel resummation for selected cases.

We have found remarkable consistency and stability of the results in $d=2$ and 3 with expected growth of deviations for a large number of arms of one star. The same methods were applied previously to the problem of uniform star polymers and have led to results [32,72] in good agreement with Monte Carlo (MC) simulations [36,37]. We hope our present calculations might also stimulate MC studies of the copolymer star problem.

The study we performed for two dimensions might have no direct application to the physics of real polymers, but it could perhaps give some insight to the problem of mapping our theory to a two-dimensional conformal field theory. The resummed values of the exponents for stars of mutually avoiding walks are in fair agreement with an exact result previously conjectured [11,12]. The exponents for the case of stars of two mutually avoiding sets of walks, on the other hand, show a dependence on the numbers of walks that may not be described by a second-order polynomial as derived from the general Kac formula [73–75]. This may be seen already qualitatively from the fact that the curvature of the function $\eta_{f_1 f_2}$ (see Fig. 5) of the two variables f_1, f_2 along each of the axes in the f_1-f_2 plane has the opposite sign to the curvature along the diagonal $f_1=f_2$.

It is this fact that shows that the series of exponents $\eta_{f_1 f_2}$ is a good candidate for finding its application in the theory of MF spectra [76]. The MF spectrum describing the moments of a fractal probability measure fulfills exact conditions of convexity. Deriving such a MF spectrum, however, from the scaling dimensions of a series of composite field operators is only possible if the scaling dimensions show the appropriate convexity [1]. This in fact is given for our case and the series of exponents may be related to the MF spectrum generated by harmonic diffusion near an absorbing fractal [5]. This also allows for a field-theoretic test of results for the short-distance correlations on multifractals [10]. This relation and the calculation of the MF spectrum on the basis of the results presented here are subjects of a separate work [38].

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APPENDIX A: LOOP INTEGRALS—GRAPHS, ε EXPANSION, AND NUMERICAL VALUES

This appendix is devoted to the contributions to perturbation theory, their representation in terms of Feynman graphs and their corresponding loop integrals, and the evaluation of these integrals for the two RG approaches. Figure 6 shows the Feynman graphs up to third loop order representing the contributions to the functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ (we keep the labeling of [56]).

Each contribution to $\Gamma^{(*f)}$ contains the composite operator $\Pi_{i=1}^{f}\phi_{a_i}$ only once. The relevant graphs can be obtained

TABLE VIII. Values of the loop integrals. The graphs 2-m1–5-s3 indicate the derivative of the two-point function $\partial/\partial k^2\Gamma^{(2)}(k)$.

Graph	Integral value	Graph	Integral value	Graph	Integral value
2-u2	D_2	8-u4	I_4	14	I_1D_2
3-u3	D_2^2	9-u4	I_5	2-m1	0
4-u3	I_1	10-u	I_6	3-s2	I_2
5-u4	D_2^3	11-u4	I_5	4-m3	0
6-u4	I_1D_2	12-u4	I_7	5-s3	I_8
7-u4	I_3	13	D_2^3		

from the usual four-point graphs $2-u2-12-u4$ by considering each vertex in turn to describe the composite operator. In the three-loop approximation we consider here two more graphs contribute that cannot be produced in this manner. They are labeled 13 and 14. In Table VIII we show the correspondence between the numerical values of the loop integrals and appropriate Feynman graphs.

A diagram with L loops is to be multiplied by σ_d^L with

$$\sigma_d=\frac{1}{2^{d-1}(\pi)^{d/2}\Gamma(d/2)},$$

but this factor can be absorbed by redefinition of the coupling constants $g_{ab}\to g_{ab}/\sigma_d$. In the massless renormalization scheme loop integrals corresponding to these graphs are evaluated by the ε expansion at zero mass and nonzero external momenta chosen at the so-called symmetry point. The expressions read [39]

$$D_2^\varepsilon=\frac{1}{\varepsilon}\left(1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2}\right),\quad I_1^\varepsilon=\frac{1}{2\varepsilon^2}\left(1+\frac{3\varepsilon}{2}+\frac{5\varepsilon^2}{2}-\frac{J\varepsilon^2}{2}\right),$$

$$I_2^\varepsilon=-\frac{1}{8\varepsilon}\left(1+\frac{5\varepsilon}{4}\right),\quad I_3^\varepsilon=-\frac{1}{24\varepsilon^2}\left(1+\frac{15\varepsilon}{4}\right),$$

$$I_4^\varepsilon=\frac{1}{6\varepsilon^3}\left(1+3\varepsilon+\frac{31\varepsilon^2}{4}-\frac{3J\varepsilon^2}{2}\right),$$

$$I_5^\varepsilon=\frac{1}{3\varepsilon^3}\left(1+\frac{5\varepsilon}{2}+\frac{23\varepsilon^2}{4}-\frac{3J\varepsilon^2}{2}\right),$$

$$I_6^\varepsilon=\frac{1}{3\varepsilon^3}\left(1+2\varepsilon+\frac{13\varepsilon^2}{4}\right),$$

$$I_7^\varepsilon=\frac{\zeta(3)}{2\varepsilon},\quad I_8^\varepsilon=-\frac{1}{6\varepsilon^2}(1+2\varepsilon).\tag{A1}$$

Here the values of derivatives $\partial/\partial k^2$ of the function $\Gamma^{(2)}(k)$ are given at the point $k^2=1$. In the massive renormalization scheme loop integrals are calculated at non-zero mass and zero external momenta [to distinguish from Eq. (A1) we will label them by m]. The mass renormalization introduces a higher-order correction to the propagator, which has to be taken into account in our calculation only in the first-order term [see Eqs. (16) and (17)]

$$D_2^{a_1a_2}=D_2+\frac{1}{9}I_2D_{21}(u_{a_1a_1}^2+u_{a_2a_2}^2).$$

Here $D_{21}=(4-d)/4$. This value has been substituted into the results for the β functions and fixed points. D_{21} does not enter expressions that are independent of the RG scheme, such as the resulting exponents. The integrals can be either ε expanded [see formulas (42) from this article for instance] or numerically calculated at arbitrary space dimensions [56,57]. In particular, for dimensions $d=2$ and 3 they are given in Table IX with the normalization

$$i_1=I_1^m/(D_2^m)^2,\quad i_2=I_2^m/(D_2^m)^2,\quad i_j=I_j^m/(D_2^m)^3,$$

$$j=3,\ldots,8.$$

Note that in the massive scheme the values of the derivative $\partial/\partial k^2$ of the function $\Gamma^{(2)}(k)$ are given at $k^2=0$.

APPENDIX B: THREE LOOP-CONTRIBUTIONS

In this appendix we have collected the more lengthy expressions for the three loop contributions to RG functions and exponents. The coefficients b^{jkl} $(j+k+l=3)$ for the τ expansion of the function $\beta_{v_{12}}^m$ (37) read

$$b^{300}=b^{030}=-\frac{i_2d}{18}-\frac{2i_3}{9}-\frac{16i_4}{9}-\frac{i_6}{3}-\frac{4i_8}{9}+\frac{58i_1}{27}+\frac{2i_2}{3}-\frac{20}{27},$$

TABLE IX. Numerical values of normalized loop integrals i_j calculated in the massive field-theory framework [56,57].

d	i_1	i_2	i_3	i_4
2	0.781302412896	-0.114635746230	-0.044703881514	0.569829439192
3	0.6666666667	-0.0740740741	-0.0376820725	0.3835760966
d	i_5	i_6	i_7	i_8
2	0.659043562065	0.650899895132	0.40068563	-0.157398409771
3	0.5194312413	0.5000000000	0.1739006107	-0.0946514319

$$b^{003}=-\frac{8i_4}{9}-\frac{2i_6}{9}-\frac{2i_7}{9}-\frac{14}{27}+\frac{34i_1}{27},$$

$$b^{210}=b^{120}=0,$$

$$b^{102}=b^{012}=-\frac{20i_4}{9}-\frac{4i_7}{9}-\frac{2i_6}{9}-\frac{26}{27}+\frac{70i_1}{27},$$

$$b^{201}=b^{021}=-\frac{2i_3}{9}-\frac{16i_4}{9}-\frac{2i_7}{9}-\frac{i_2d}{18}-\frac{28}{27}+\frac{20i_1}{9}+\frac{2i_2}{27},$$

$$b^{111}=-\frac{16}{27}-\frac{2i_6}{9}-\frac{8i_4}{9}+\frac{4i_1}{3}.$$

The coefficients $\eta_{U;k_1;k_2}$ introduced for the τ expansion of the exponent $\eta^U_{f_1f_2}$ in the unsymmetric fixed point U (58) read

$$\eta_{U;0;0}=328-1480i_1-128i_2-240i_4-492i_5+132i_6-356i_7-48i_8+2288i_1^2+16i_2^2+384i_1i_2,$$

$$\eta_{U;0;1}=-7680i_1^2-496i_1i_2+5708i_1+184i_2+1326i_7+1620i_5-588i_6+204i_4-16i_2^2+48i_8-1312,$$

$$\eta_{U;0;2}=570+810i_4+1488i_1^2-24i_2+48i_1i_2-2154i_1-270i_7+216i_6-216i_5,$$

$$\eta_{U;0;3}=-54-162i_4+162i_1,$$

$$\eta_{U;1;0}=-16i_2^2+468i_7-448i_1i_2+160i_2+2408i_1-252i_6-2992i_1^2+564i_5+48i_8-560,$$

$$\eta_{U;1;1}=-5342i_1+176i_1i_2-88i_2+756i_6-594i_7-252i_5+3536i_1^2+1638i_4+1346,$$

$$\eta_{U;1;2}=1188i_1-216i_5-216i_6-756i_4-324,$$

$$\eta_{U;2;0}=-1072i_1-32i_2+64i_1i_2+336i_4-112i_7+144i_6+704i_1^2-48i_5+272,$$

$$\eta_{U;2;1}=1098i_1-180i_6-738i_4-180i_5-306,$$

$$\eta_{U;3;0}=-40+144i_1-24i_6-96i_4-24i_5.$$

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